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Discrete orthogonal projections on multiple knot periodic splines

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Abstract

This paper establishes properties of discrete orthogonal projections on periodic spline spaces of order r, with knots that are equally spaced and of arbitrary multiplicity $M \leqslant r$. The discrete orthogonal projection is expressed in terms of a quadrature rule formed by mapping a fixed J-point rule to each sub-interval. The results include stability with respect to discrete and continuous norms, convergence, commutator and superapproximation properties. A key role is played by a novel basis for the spline space of multiplicity M, which reduces to a familiar basis when $M = 1$.

1. Introduction

Periodic splines have a long history. Periodic splines are 1-periodic real-valued functions, which are piecewise polynomials of order r (i.e. of degree at most $r-1$ on each subinterval), with equally spaced knots $x_j = jh = j/N$ for $j \in \mathbb{Z}$, and most often with maximum smoothness (i.e. C^{r-2}) at each knot. Early papers devoted to smoothest periodic splines are [5,11,14]. The difference in this paper is that we allow knots of arbitrary (but constant) multiplicity M; equivalently, we require only C^{r-M-1} continuity at the knots. Splines with multiple knots have previously been studied in [8].

More precisely, we consider 1-periodic splines, i.e. splines on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, with knots of equal multiplicity $M \geqslant 1$, on equidistant meshes. Let r, M, N with $M \leqslant r$ be positive integers.

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We define the set of knots

$$\pi_h := \{x_j = jh,\ j = 0, \ldots, N-1\} \quad \text{for } h \in \mathcal{H} := \{1/N,\ N \in \mathbb{N}\}$$

and denote by $S_h^{r,M}$ the space of 1-periodic splines of order r (i.e. piecewise polynomials of degree at most $r-1$) with M-fold breakpoints at the knots in π_h. It is easily seen that $S_h^{r,M}$ is a subspace of C^{r-M-1} of dimension MN, where $C^k = C^k(\mathbb{T})$ is the space of 1-periodic k times continuously differentiable functions. Here, C^{-1} means that jumps are allowed at the knots in π_h, where the value of the function at a point of discontinuity is normalised to be the arithmetic mean of the left- and right-hand limits.

In the present paper, we study the approximation of a 1-periodic function (or distribution) f by a periodic spline f_h of order $r \geqslant 1$ and knots of arbitrary multiplicity $M \in [1, r]$ by what we shall call 'discrete orthogonal projection'. Discrete orthogonal projection is based on a composite quadrature rule

$$Q_N f := h \sum_{k=0}^{N-1} \sum_{j=1}^{J} \omega_j f(x_{k,j}), \quad x_{k,j} := x_k + h\xi_j, \tag{1}$$

derived from the basic quadrature formula

$$Q_1 f := \sum_{j=1}^{J} \omega_j f(\xi_j), \tag{2}$$

where the quadrature points $\{\xi_j\}$ and weights $\{\omega_j\}$ satisfy

$$0 \leqslant \xi_1 < \xi_2 < \cdots < \xi_J < 1, \quad J \geqslant M, \quad \sum_{j=1}^{J} \omega_j = 1, \quad \omega_j > 0. \tag{3}$$

Associated with the quadrature rule is an inner product

$$(v_h, w_h)_h := Q_N(v_h \overline{w}_h) \tag{4}$$

on the linear space W_h of 'grid' functions v_h and w_h, which are functions defined on the grid of quadrature points

$$\pi'_h := \{x_{k,j} = x_k + h\xi_j,\ k = 0, \ldots, N-1,\ j = 1, \ldots, J\}. \tag{5}$$

The inner product in (4) can be thought of as an approximation to

$$(v, w) := \int_0^1 v(x)\overline{w}(x)\, dx \quad \text{for } v, w \in L^2(\mathbb{T}). \tag{6}$$

Given the grid function f_h, the discrete orthogonal projection $R_h f_h$ is the spline defined by

$$R_h f_h \in S_h^{r,M} \quad (R_h f_h, z_h)_h = (f_h, z_h)_h \quad \text{for all } z_h \in S_h^{r,M}. \tag{7}$$

Conditions for this to be a valid definition are given in Section 3. Clearly, $R_h f_h$ can be viewed as a discrete version of the orthogonal projection $O_h f$ defined by

$$O_h f \in S_h^{r,M} \quad (O_h f, z_h) = (f, z_h) \quad \text{for all } z_h \in S_h^{r,M}.$$

For the particular case $J = M$ it is easily seen that discrete orthogonal projection is equivalent to interpolation: that is,

$$(R_h f_h)(x_{k,j}) = f_h(x_{k,j}) \quad \text{for } k = 0, \ldots, N-1, \ j = 1, \ldots, M.$$

The significance of the periodicity of the splines and the equal separation of their knots is that a much more refined (Fourier) analysis is available for periodic splines than for splines with general meshes. For this reason, the analysis in this paper is carried out in Sobolev spaces of arbitrary real order s (see for example [12, Chapter 5]). For $s \in \mathbb{R}$, let $H^s = H^s(\mathbb{T})$ denote the usual Sobolev space of periodic distributions equipped with the norm

$$\|f\|_s := \left(\sum_{n=-\infty}^{\infty} \langle n \rangle^{2s} |\hat{f}(n)|^2 \right)^{1/2} \quad \text{with } \langle n \rangle := \begin{cases} 1 & \text{if } n = 0, \\ |n| & \text{if } n \neq 0, \end{cases} \tag{8}$$

where

$$\hat{f}(n) := \int_{\mathbb{T}} f(x) e^{-i2\pi n x}\, dx \quad \text{for } n \in \mathbb{Z}$$

are the complex Fourier coefficients of the 1-periodic distribution f. It is well-known (see e.g. [8]) that for $1 \leqslant M \leqslant r$

$$S_h^{r,M} \subset H^s \quad \text{for } s < r - M + \tfrac{1}{2}.$$

The present study is motivated also by the desire to analyse spline methods for more general operator equations of the form

$$Lu = f,$$

where L is a pseudo-differential operator. Such an equation may, for example, represent a boundary integral equation on a smooth curve, after appropriate parameterisation. The earliest such application, for the smoothest spline case, seems to have been that of Quade and Collatz [11], who used Fourier techniques to analyse the collocation approximation with C^0 piecewise linear splines (i.e. $r = 2$ and $M = 1$) for certain pseudo-differential operator equations. For a smoothest spline of general order r, the collocation approximation has the form

$$Lu_h(jh + \varepsilon h) = f(jh + \varepsilon h) \quad \text{for } j = 0, \ldots, N-1,$$

where u_h is a spline of order r, and ε is a number that determines the location of the collocation points in relation to the knots $\{jh\}$. (In the Quade and Collatz case $\varepsilon = 0$.) Note especially that in the case of smoothest splines there is only one degree of freedom per interval: the dimensionality of the space of smoothest splines is just N. The simplicity resulting from just one degree of freedom per interval is lost when multiple knots are allowed. Analyses of collocation methods for more general operator equations, and for smoothest splines of arbitrary orders, have been made by Arnold [1], Arnold and Wendland [2] and Saranen and Wendland [13]. Collocation for operator equations with multiple knots were studied by McLean and Prößdorf [8].

Another smoothest spline method for operator equations is the 'qualocation approximation' [4,16,18]; for a review see [15]. This takes the form

$$(Lu_h, z_h)_h = (f, z_h)_h \quad \text{for all } z_h \in S_h^{r,M}.$$

Clearly, approximation methods of this kind are closely related to discrete orthogonal projection; indeed, discrete orthogonal projection is the special case in which L is the identity operator. The qualocation method is equivalent to collocation when $J = M$.

An important tool in the whole paper is a new spline basis introduced in Section 2, which may be of interest in its own right. This basis is also suitable for performing concrete numerical calculations. The new basis is a consistent extension of one first introduced in [11,14] for splines with simple knots. It contrasts with the recursive characterisation of multiple knot splines used in [8].

Section 3 is devoted to stability properties and the approximation power of the spline projection R_h. It turns out to be very helpful for the analysis if these properties are established, as they are in Proposition 3.11, also with respect to the norm $\|\cdot\|_h$, defined in the space of grid functions W_h by

$$\|f_h\|_h := (f_h, f_h)_h^{1/2} \quad \text{for } f_h \in W_h.$$

A helpful result is the norm equivalence result on spline spaces given in Proposition 3.3.

As is well known, the analysis of variable coefficient operators L relies on 'superapproximation' and 'commutator' properties of the spline spaces. (In [9], these properties are called commutator properties of types I and II, and labelled as CPI, CPII.) For the case of smoothest splines these properties were proved in [7]. For the multiple-knot case they are proved here as Theorems 4.1 and 4.4. At the same time various results in the literature are sharpened, and sometimes proved in a simpler way. The proof of the so-called dual commutator property in Theorem 4.4 relies essentially on stability and approximation with respect to the norm $\|\cdot\|_h$.

As an example, we show in Section 5 how our results can be applied to interpolation for splines with double knots. Recall that interpolation is the special case of discrete orthogonal projection when the number of quadrature points per interval is equal to the number of degrees of freedom, i.e. in this case $J = M = 2$. A conjecture left open in [8] for the case of double knots is proved here as Proposition 5.2.

A set of basic formulas used throughout the paper is collected in the Appendix.

2. A spline basis

A key element for the analysis in this paper is a suitable basis for the spline space $S_h^{r,M}$, the space of periodic splines of order r with knots of multiplicity M. In the collocation analysis in [8] the authors used a recursive characterisation of such splines. We owe much of the progress in this paper to working with a basis that generalises the nice basis for the case $M = 1$ in [4,5,14]. For this purpose we define for $M \leqslant r$ the functions

$$\tilde{\Delta}_k(\xi, y) := \sum_{\ell \neq 0} \frac{\ell^{k-1}}{(y+\ell)^r} \Phi_\ell(\xi) \quad \text{for } |y| \leqslant \frac{1}{2}, \xi \in \mathbb{R} \text{ and } k = 1, \dots, M, \tag{9}$$

$$\Phi_\ell(\xi) := \exp(i2\pi\ell\xi) \quad \text{for } \ell \in \mathbb{Z} \text{ and } \xi \in \mathbb{R}, \tag{10}$$

$$\Delta_1(\xi, y) := 1 + y^r \tilde{\Delta}_1(\xi, y), \tag{11}$$

$$\Delta_k(\xi, y) := \tilde{\Delta}_k(\xi, y) \quad \text{for } k = 2, \dots, M, \tag{12}$$

$$\psi_{k,\mu}(x) := \Phi_\mu(x) \Delta_k\left(Nx, \frac{\mu}{N}\right) \quad \text{for } k = 1, \dots, M \text{ and } \mu \in \Lambda_h, \tag{13}$$

where

$$\Lambda_h := \left(-\frac{N}{2}, \frac{N}{2}\right] \cap \mathbb{Z} \quad \text{for } Nh = 1. \tag{14}$$

The Fourier series in (9) may not be absolutely convergent. We always understand the sum to mean the limit as $L \to \infty$ of the symmetric partial sums extended from $-L$ to L. In the case $M = 1$ we obtain the basis used in [4].

Proposition 2.1. *The set $\psi_{k,\mu}$ for $k = 1, \ldots, M$ and $\mu \in \Lambda_h$ is a basis in $S_h^{r,M}$.*

Proof. We first show that $\psi_{k,\mu} \in S_h^{r,M}$ for $k = 1, \ldots, M$. It is known from [4] that $\psi_{1,\mu} \in S_h^{r,1} \subset S_h^{r,M}$. It is then clear that also the derivatives $\psi_{1,\mu}^{(j)} \in S_h^{r-j,1} \subset S_h^{r,M}$ for $j = 1, \ldots, M-1$ (here in the case $M = r$ the value of $\psi_{1,\mu}^{(M-1)}(x)$ at points $x \in \pi_h$ is defined to be the arithmetic mean of its one-sided limits). If $M = 1$ we are done, so assume $M > 1$. We prove that for $k = 2, \ldots, M$ each $\psi_{k,\mu}$ is a linear combination of these functions.

Note first that for y fixed, $\Delta_1(\cdot, y)$ is $(M-1)$-fold differentiable because $M \leqslant r$, except at the points of π_h (see [3]), which can be excluded in the following consideration. The differentiation can be performed term by term, yielding

$$\frac{d^j \Delta_1(Nx, y)}{dx^j} = (i2\pi N)^j y^r \Delta_{j+1}(Nx, y) \quad \text{for } j = 1, \ldots, M-1.$$

By the Leibniz rule we obtain for $k = 1, \ldots, M-1$,

$$\begin{aligned} \psi_{1,\mu}^{(k)}(x) &= \sum_{j=0}^{k} \binom{k}{j} \Phi_\mu^{(k-j)}(x) \, \frac{d^j \Delta_1\left(Nx, \frac{\mu}{N}\right)}{dx^j} \\ &= (i2\pi\mu)^k \Phi_\mu(x) \Delta_1\left(Nx, \frac{\mu}{N}\right) \\ &\quad + \sum_{j=1}^{k} \binom{k}{j} (i2\pi\mu)^{k-j} \Phi_\mu(x) (i2\pi N)^j \left(\frac{\mu}{N}\right)^r \Delta_{j+1}\left(Nx, \frac{\mu}{N}\right) \\ &= (i2\pi\mu)^k \psi_{1,\mu}(x) + \left(\frac{\mu}{N}\right)^r \sum_{j=1}^{k} \binom{k}{j} (i2\pi\mu)^{k-j} (i2\pi N)^j \psi_{j+1,\mu}(x). \end{aligned} \tag{15}$$

In the case $\mu \neq 0$ it is seen successively from (15) that $\psi_{k+1,\mu} \in S_h^{r,M}$ for $k = 1, \ldots, M-1$.

We are left with the case $\mu = 0$. Since $\psi_{1,0} = 1$ we obtain no information by differentiation. But note that

$$\psi_{k,0}(x) = \Delta_k(Nx, 0) = \sum_{j \neq 0} j^{k-1-r} \Phi_{jN}(x) \quad \text{for } k = 2, \ldots, M.$$

The sum

$$\sum_{j \neq 0} j^{-1} \Phi_{jN}(x)$$

is the Fourier series of the h-periodically extended piecewise continuous function $f(x) = i2\pi(1/2 - Nx), x \in (0, h), f(0) := 0$. It therefore belongs to $S_h^{2,2}$. Term by term integration $r - k$ times shows that $\psi_{k,0} \in S_h^{r-(k-2),2} \subset S_h^{r,M}$.

We turn now to the proof of the linear independence of the functions $\{\psi_{k,\mu}, \mu \in \Lambda_h, k = 1, \ldots, M\}$. It is shown in Lemma A.1 that the Gram matrix of this set is block diagonal with blocks labelled by μ, thus it is sufficient to prove that for each fixed μ the set $\{\psi_{k,\mu}, k = 1, \ldots, M\}$ is linearly independent. For $\mu = 0$ we have seen before that the $\{\psi_{k,0}, k = 1, \ldots, M\}$ are piecewise polynomials of different degrees, which are consequently linearly independent. For $\mu \neq 0$ the recursion (15) yields the representation

$$\psi_{k,\mu} = \sum_{j=0}^{k-1} c_{k,j}\, \psi_{1,\mu}^{(j)} \quad \text{for } k = 2, \ldots, M \tag{16}$$

with $c_{k,k-1} \neq 0$. From the Fourier series of $\psi_{1,\mu}$ it is known that $\psi_{1,\mu}$ is a piecewise polynomial of exact degree $r-1$ on each subinterval, thus the successive derivatives $\psi_{1,\mu}^{(j)}$ all have different degrees, implying the linear independence we wanted to prove. □

The following result exposes the essential nature of the basis $\{\psi_{k,\mu}\}$ in the spline space $S_h^{r,M}$: under translation by h each such spline behaves like the complex exponential function Φ_μ; while for fixed $\mu \in \Lambda_h$ the sequence $\psi_{1,\mu}, \psi_{2,\mu}, \ldots, \psi_{M,\mu}$ has decreasing smoothness, with the multiplicity of the knots increasing from one member of the sequence to the next.

Proposition 2.2. *For $\mu \in \Lambda_h$ and $k = 1, \ldots, M$*

$$\psi_{k,\mu}(x+h) = e^{2\pi i \mu h} \psi_{k,\mu}(x),$$
$$\psi_{k,\mu} \in S_h^{r,k}.$$

Proof. The first result is trivial on noting $\Delta(\xi + 1, y) = \Delta(\xi, y)$. For $\mu \neq 0$ the second follows from (16), on using

$$\psi_{1,\mu}^{(k-1)} \in S_h^{r-k+1,1} \subset S_h^{r,k}.$$

For $\mu = 0$ we noted already in the proof of Proposition 2.1 that

$$\psi_{k,0} \in S_h^{r-k+2,2} \subset S_h^{r,k} \quad \text{for } k = 2, \ldots, M,$$

while $\psi_{1,0} = 1 \subset S_h^{r,1}$. □

The new basis can be derived systematically by Fourier transformation of the B-spline basis in the usual divided difference form.

3. The discrete orthogonal projection R_h

The projection $R_h : W_h \to S_h^{r,M}$ is defined by

$$(R_h f_h, \psi)_h = (f_h, \psi)_h \quad \text{for all } \psi \in S_h^{r,M}. \tag{17}$$

In this section we give conditions for R_h to be well defined, and study some important properties of R_h.

Lemma 3.1. *The positive semidefinite sesquilinear form* $(\cdot,\cdot)_h$ *is an inner product on* $S_h^{r,M}$, *and hence* R_h *is well defined, iff for all* $\mu \in \Lambda_h$ *the M functions* $\{\Delta_k(\cdot, \frac{\mu}{N})), k = 1, \ldots, M\}$ *are linearly independent on the set of quadrature points* $\{\xi_j,\ j = 1, \ldots, J\}$.

Proof. It follows from Lemma A.3 that the Gram matrix with respect to $(\cdot,\cdot)_h$ of the spline basis $\{\psi_{k,\mu}\}$ is block diagonal, where the elements of the block matrices $B(y)$, $y = \mu/N$, $\mu \in \Lambda_h$ are given by

$$B_{k,\ell}(y) := Q\left(\Delta_k(\cdot, y), \Delta_\ell(\cdot, y)\right) \quad \text{for } k, \ell = 1, \ldots, M \tag{18}$$

with

$$Q(v, w) := Q_1(v\overline{w}) = \sum_{j=1}^{J} \omega_j v(\xi_j)\overline{w}(\xi_j). \tag{19}$$

Since Q is a positive semidefinite sesquilinear form on grid functions on the quadrature points, the assertion follows immediately. $\square$

Remark 3.2. Clearly, $J \geqslant M$ is necessary for R_h to be well defined. In the case $J = M$ note that $R_h f_h$ is nothing else than interpolation at the quadrature points. For $J > M$ a consequence of Lemma 3.1 is that the projection R_h is well defined iff there exists a selection of M quadrature points from (3), such that the corresponding interpolatory projection is well defined.

The following proposition, an equivalence property of two norms for the space $S_h^{r,M}$, plays an important role in the analysis.

Proposition 3.3. *Let* $1 \leqslant M \leqslant r$ *and* $s < r - M + \frac{1}{2}$. *On* $S_h^{r,M}$ *the norm* $\|\cdot\|_s$ *is equivalent, uniformly for* $h \in \mathcal{H}$, *to the norm*

$$\|v_h\|_{s,h} := \left(\sum_{\mu\in\Lambda_h} \left[\langle\mu\rangle^{2s} |c_{1,\mu}|^2 + N^{2s} \sum_{k=2}^{M} |c_{k,\mu}|^2 \right] \right)^{1/2}, \tag{20}$$

where the coefficients $c_{k,\mu}$ *are defined by the unique representation*

$$v_h = \sum_{k=1}^{M} \sum_{\mu\in\Lambda_h} c_{k,\mu}\psi_{k,\mu} \quad \textit{for } v_h \in S_h^{r,M}. \tag{21}$$

Proof. Introducing the notation used in Lemma A.6,

$$\tilde{c}_{k,\mu} := \begin{cases} \mu^r N^{s-r} c_{1,\mu} & \text{for } k = 1, \\ N^s c_{k,\mu} & \text{for } k = 2, \ldots, M, \end{cases}$$

$$\Omega_k^{(s)}(\xi, y) := \sum_{\ell\neq 0} |y+\ell|^s \frac{\ell^{k-1}}{(y+\ell)^r} \Phi_\ell(\xi) \quad \text{for } \xi \in \mathbb{R} \text{ and } |y| \leqslant \frac{1}{2},$$

we can write, from (20)

$$\|v_h\|_{s,h}^2 = \sum_{\mu\in\Lambda_h} \langle\mu\rangle^{2s} |c_{1,\mu}|^2 + \sum_{\mu\in\Lambda_h} \sum_{k=2}^{M} |\tilde{c}_{k,\mu}|^2, \tag{22}$$

while from Lemma A.6

$$\|v_h\|_s^2 = \sum_{\mu\in\Lambda_h} \langle\mu\rangle^{2s}|c_{1,\mu}|^2 + \sum_{\mu\in\Lambda_h} \left\| \sum_{k=1}^{M} \tilde{c}_{k,\mu}\Omega_k^{(s)}\left(\cdot, \frac{\mu}{N}\right)\right\|_0^2. \tag{23}$$

The latter norm is associated with the sesquilinear form

$$\sum_{k=1}^{M}\sum_{\ell=1}^{M} a_k G_{k,\ell}(y)\overline{a}_\ell \tag{24}$$

on $\mathbb{C}^M$ with $y = \mu/N$ of the Gram matrix G with elements

$$G_{k,\ell}(y) := \left(\Omega_k^{(s)}(\cdot, y), \Omega_\ell^{(s)}(\cdot, y)\right)_0 \quad \text{for } |y| \leqslant 1/2.$$

The coefficients $G_{k,\ell}(\cdot)$ are continuous for $s + M - 1 - r < -\frac{1}{2}$. Hence, the sesquilinear form (24) is bounded uniformly in y, yielding

$$\left\|\sum_{k=1}^{M} \tilde{c}_{k,\mu}\Omega_k^{(s)}\left(\cdot, \frac{\mu}{N}\right)\right\|_0^2 \leqslant C\left(\sum_{k=1}^{M}|\tilde{c}_{k,\mu}|\right)^2 \leqslant C\sum_{k=1}^{M}|\tilde{c}_{k,\mu}|^2,$$

where here and elsewhere C denotes a generic constant, which may take different values at different occurrences. To prove $\|v_h\|_s \leqslant C\|v_h\|_{s,h}$ we thus estimate the second term in (23) as

$$\begin{aligned}\sum_{\mu\in\Lambda_h}\left\|\sum_{k=1}^{M} \tilde{c}_{k,\mu}\Omega_k^{(s)}\left(\cdot, \frac{\mu}{N}\right)\right\|_0^2 &\leqslant C\sum_{\mu\in\Lambda_h}\left(\frac{\mu^{2r}}{N^{2r-2s}}|c_{1,\mu}|^2 + \sum_{k=2}^{M}|\tilde{c}_{k,\mu}|^2\right)\\ &\leqslant C\sum_{\mu\in\Lambda_h}\left(\langle\mu\rangle^{2s}|c_{1,\mu}|^2 + \sum_{k=2}^{M}|\tilde{c}_{k,\mu}|^2\right) \leqslant C\|v_h\|_{s,h}^2,\end{aligned}$$

where we needed $|\mu/N| \leqslant \frac{1}{2}$, $r > s$ and $|\mu| \leqslant \langle\mu\rangle$.

To prove the inequality in the reverse direction it is sufficient to show that the sesquilinear form (24) is uniformly in y positive definite. Taking the continuity of $G(\cdot)$ into account, this follows if we prove for each fixed y that the functions $\Omega_k^{(s)}(\cdot, y)$ are linearly independent. To verify this property let

$$\sum_{k=1}^{M} c_k\Omega_k^{(s)}(\xi, y) = 0 \quad \text{for } \xi \in (0, 1).$$

After multiplication with $\Phi_\ell(\xi)$ and integration we obtain the linear system

$$\sum_{k=1}^{M} c_k|y+\ell|^s\frac{\ell^{k-1}}{(y+\ell)^r} = 0 \quad \text{for } \ell = 1, \ldots, M,$$

which has a scaled Vandermonde matrix, and hence is nonsingular, implying $c_1 = \cdots = c_M = 0$. □

In the special case $s = 0$ we deduce from Proposition 3.3 the following:

Corollary 3.4. *Let $1 \leqslant M \leqslant r$. There exist constants $0 < c < c'$ such that for all $h \in \mathcal{H}$ and $v_h \in S_h^{r,M}$ the norm equivalence*

$$c\|v_h\|_0 \leqslant \left(\sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} |c_{k,\mu}|^2 \right)^{1/2} \leqslant c'\|v_h\|_0 \tag{25}$$

holds, where v_h has the form (21).

Next we prove a norm equivalence between the continuous and discrete norms corresponding to (6) and (4), respectively.

Proposition 3.5. *Let $1 \leqslant M \leqslant r$. The norms $\|\cdot\|_0$ and $\|\cdot\|_h$ are equivalent on $S_h^{r,M}$, uniformly for $h \in \mathcal{H}$, iff the functions $\{\Delta_k(\cdot, y), k = 1, \ldots, M\}$, are for all $y \in [-\frac{1}{2}, \frac{1}{2}]$ linearly independent on the set of quadrature points $\{\xi_j\}$.*

Proof. From Corollary A.3 we have, for v_h written in the form (21), the relation

$$\|v_h\|_h^2 = \left\| \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} c_{k,\mu} \psi_{k,\mu} \right\|_h^2 = \sum_{k,\ell=1}^{M} \sum_{\mu \in \Lambda_h} B_{k,\ell}\left(\frac{\mu}{N}\right) c_{k,\mu} \overline{c}_{\ell,\mu}. \tag{26}$$

Note that the functions $\Delta_k(\xi_j, \cdot)$, $j = 1, \ldots, J$, are continuous on $[-\frac{1}{2}, \frac{1}{2}]$. Hence the elements $B_{k,\ell}(y)$ of the matrix $B(y)$ are bounded uniformly in y, and it follows from (26) and Corollary 3.4 that

$$\|v_h\|_h^2 \leqslant C \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} |c_{k,\mu}|^2 \leqslant C\|v_h\|_0^2. \tag{27}$$

We now show that the conditions in the proposition are equivalent to

$$\sum_{k=1}^{M} |a_k|^2 \leqslant C \sum_{k,\ell=1}^{M} B_{k,\ell}\left(\frac{\mu}{N}\right) a_k \overline{a}_l \quad \text{for } h \in \mathcal{H}, \mu \in \Lambda_h, a \in \mathbb{C}^M, \tag{28}$$

which in view of Corollary 3.4 completes the proof. Assume first that the $\Delta_k(\cdot, y)$ are linearly independent on the set of quadrature points. Then $B(y)$ as a Gram matrix is positive definite for each $y \in [-\frac{1}{2}, \frac{1}{2}]$. From the continuity of $B(\cdot)$ we conclude that it is positive definite uniformly in y and (28) follows. In the reverse direction, it follows from (28) that this inequality holds with μ/N replaced by $y \in [-\frac{1}{2}, \frac{1}{2}]$ since the set $\{\mu/N, \mu \in \Lambda_h, h \in \mathcal{H}\}$ is dense there. Thus, the Gram matrix $B(y)$ is positive definite and consequently the $\Delta_k(\cdot, y)$ are linearly independent. □

We now turn to the stability and approximation power of R_h. We will frequently use the following definition.

Definition 3.6. We say that the condition (R) is satisfied if the functions $\{\Delta_k(\cdot, y), k = 1, \ldots, M\}$ are linearly independent on the set of quadrature points $\{\xi_j, j = 1, \ldots, J\}$ for all $|y| \leqslant \frac{1}{2}$.

Our next aim is to establish approximation properties for the sequence $\{R_h\}_{\mathcal{H}}$. For this we need to know the approximation power of the spline spaces $S_h^{r,M}$, $h \in \mathcal{H}$. For this purpose we can use the projection $P_h : H^t \to S_h^{r,M}$ which was introduced in [8, p. 428] through the definition

$$P_h f \in S_h^{r,M} \quad (P_h f, \Phi)_0 = (f, \Phi)_0 \quad \text{for } \Phi \in S_h^{\infty,M}, \tag{29}$$

where

$$S_h^{\infty,M} := \operatorname{span}\{\Phi_{\mu+\ell N}, \mu \in \Lambda_h, \ell \in (-M/2, M/2]\}.$$

It is shown in [8, Theorem 3.4] that for $s < r - M + \frac{1}{2}$ and $s \leqslant t \leqslant r$

$$\|P_h f - f\|_s \leqslant C h^{t-s} \|f\|_t \quad \text{for } f \in H^t. \tag{30}$$

Proposition 3.7. *Let condition* (R) *be satisfied and assume* $0 \leqslant s < r - M + \frac{1}{2}$, $s \leqslant t \leqslant r$ *and* $\frac{1}{2} < t$. *Then*

$$\|R_h f - f\|_s \leqslant C h^{t-s} \|f\|_t \quad \textit{for } f \in H^t. \tag{31}$$

Proof. Let P_h be the projection from (29). We show that

$$\|R_h f - P_h f\|_0 \leqslant C h^t \|f\|_t \quad \text{for } f \in H^t. \tag{32}$$

The assertion then follows, after an application of the inverse property for spline spaces (see [8, Theorem 3.4]), from (30) and the triangle inequality.

For the proof of (32) we first note that the Fourier coefficients of $P_h f$ satisfy (see [8, p. 428])

$$\widehat{P_h f}(\mu) = \hat{f}(\mu) \quad \text{for } \mu \in \Lambda_h,$$

and, consequently, in the Fourier expansion of

$$g := f - P_h f = \sum_{\ell \in \mathbb{Z}} \hat{g}(\ell) \Phi_\ell,$$

we have $\hat{g}(\ell) = 0$ for $\ell \in \Lambda_h$. The system determining the coefficients $c_{k,\mu}$ in

$$R_h f - P_h f = R_h g = \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} c_{k,\mu} \psi_{k,\mu}$$

(note that $R_h P_h = P_h$ from the projection property of R_h) is block-diagonal, with the μth block given by

$$B\left(\frac{\mu}{N}\right) c_\mu = d_\mu \quad \text{for } \mu \in \Lambda_h,$$

where $c_\mu \in \mathbb{C}^M$ is the vector with components $c_{k,\mu}, k = 1, \ldots, M$, and d_μ has components $d_{k,\mu} = (g, \psi_{k,\mu})_h, k = 1, \ldots, M$. We denote by $|\cdot|$ the Euclidean norm in $\mathbb{C}^M$. Now $B(y)$, the matrix defined in (18), has a uniformly in y bounded inverse, thus on taking the norm equivalence (25) into account we obtain

$$\|R_h g\|_0^2 \leqslant C \sum_{\mu \in \Lambda_h} |c_\mu|^2 \leqslant C \sum_{\mu \in \Lambda_h} |d_\mu|^2. \tag{33}$$

With the aid of Corollary A.4 we obtain

$$d_{k,\mu} = (g, \psi_{k,\mu})_h = \sum_{\ell\in\mathbb{Z}} \hat{g}(\ell)(\Phi_\ell, \psi_{k,\mu})_h$$
$$= \sum_{\ell\equiv\mu} \hat{g}(\ell) Q\left(\Phi_{\frac{\ell-\mu}{N}}, \Delta_k\left(\cdot, \frac{\mu}{N}\right)\right). \tag{34}$$

Thus on recalling $\hat{g}(\ell) = 0$ for $\ell \in \Lambda_h$,

$$|d_\mu|^2 = \sum_{k=1}^{M} |(g, \psi_{k,\mu})_h|^2 \leqslant C \left(\sum_{\ell\equiv\mu}' |\hat{g}(\ell)|\right)^2$$
$$= C\left(\sum_{\ell\neq 0} |\mu+\ell N|^{-t}|\mu+\ell N|^t|\hat{g}(\mu+\ell N)|\right)^2$$
$$\leqslant C \sum_{\ell\neq 0} |\mu+\ell N|^{-2t} \sum_{\ell\neq 0} |\mu+\ell N|^{2t}|\hat{g}(\mu+\ell N)|^2$$
$$\leqslant Ch^{2t} \sum_{\ell\neq 0} \left|\frac{\mu}{N}+\ell\right|^{-2t} \sum_{\ell\neq 0} |\mu+\ell N|^{2t}|\hat{g}(\mu+\ell N)|^2. \tag{35}$$

Since $2t > 1$, the sum $\sum |y+\ell|^{-2t}$ converges uniformly for $|y| \leqslant 1/2$, and since also

$$\sum_{\mu\in\Lambda_h} \sum_{\ell\neq 0} |\mu+\ell N|^{2t}|\hat{g}(\mu+\ell N)|^2 \leqslant \|g\|_t^2 \leqslant C\|f\|_t^2,$$

where for the last inequality (30) was used with $t = s$, a summation of (35) with respect to $\mu \in \Lambda_h$ delivers (32). □

Corollary 3.8. *If condition* (R) *is satisfied and* $1 \leqslant M < r$ (*and hence* $r \geqslant 2$) *the following stability estimate holds for* $\frac{1}{2} < s < r - M + \frac{1}{2}$:

$$\|R_h f\|_s \leqslant C\|f\|_s \quad \textit{for } f \in H^s. \tag{36}$$

With respect to the norm $\|\cdot\|_h$ the projection R_h has the same approximation power as with respect to the norm $\|\cdot\|_0$. As a preparation for the proof of this fact we provide two lemmas.

Lemma 3.9. *Let* $\frac{1}{2} < \sigma \leqslant 1$. *Then*

$$\|f\|_h \leqslant C\left(\|f\|_0 + h^\sigma\|f\|_\sigma\right) \quad \textit{for } f \in H^\sigma. \tag{37}$$

Proof. We consider first the case $\sigma \in (\frac{1}{2}, 1)$. Let $W^{0,\sigma}(0,1)$ denote the usual fractional-order Sobolev space (see, for example, [10]) equipped with the norm

$$\|g\|_{0,\sigma} := \left(\|g\|_0^2 + \int_0^1\int_0^1 \frac{|g(\tau)-g(\eta)|^2}{|\tau-\eta|^{1+2\sigma}}\, d\tau\, d\eta\right)^{1/2}. \tag{38}$$

Let $f \in H^\sigma$ be given. For $x_n \in \pi_h$ we define $g_n \in W^{0,\sigma}(0,1)$ by

$$g_n(\xi) := f(x_n + h\xi) \quad \text{for } \xi \in [0,1].$$

Since the imbedding $W^{0,\sigma}(0,1) \hookrightarrow C[0,1]$ is continuous, there exists C such that

$$\begin{aligned} |g_n(\xi)|^2 &\leqslant C\|g_n\|_{0,\sigma}^2 \\ &= C\left(h^{-1}\|f\|_{L^2(I_n)}^2 + h^{2\sigma-1}\int_{I_n}\int_{I_n}\frac{|f(x)-f(y)|^2}{|x-y|^{1+2\sigma}}\,dx\,dy\right) \\ &\leqslant C\left(h^{-1}\|f\|_{L^2(I_n)}^2 + h^{2\sigma-1}\int_{I_n}\int_0^1\frac{|f(x)-f(y)|^2}{|x-y|^{1+2\sigma}}\,dx\,dy\right), \end{aligned} \tag{39}$$

where $I_n := (x_n, x_{n+1})$, and an obvious variable substitution was used to obtain the equality in the second step. Choose $\xi = \xi_j$, multiply by $h\omega_j$ and sum with respect to j and n to arrive at

$$\|f\|_h^2 \leqslant C\sum_{j=1}^J \omega_j\left(\|f\|_0^2 + h^{2\sigma}\int_0^1\int_0^1\frac{|f(x)-f(y)|^2}{|x-y|^{1+2\sigma}}\,dx\,dy\right).$$

The assertion now follows since the norms $\|\cdot\|_{0,\sigma}$ and $\|\cdot\|_\sigma$ are equivalent on H^σ. The assertion in the case $\sigma = 1$ can be obtained by using instead of $W^{0,\sigma}(0,1)$ the space $H^1(0,1)$ and instead of (38) the derivative definition of the H^1 norm, and by using in (39) the continuity of the imbedding $H^1 \hookrightarrow C[0,1]$. □

Lemma 3.10. *Let* $\frac{1}{2} < \sigma \leqslant 1$. *Then*

$$\|f - f_h\|_h \leqslant C\left(\|f - f_h\|_0 + h^\sigma\|f\|_\sigma\right) \quad \text{for } f \in H^\sigma \text{ and } f_h \in S_h^{1,1}. \tag{40}$$

Proof. We define the function g_n as in the proof of the last lemma with f replaced by $f - f_h$ and derive in the same way as there in place of (39) the inequality

$$|g_n(\xi)|^2 \leqslant C\left(h^{-1}\|f - f_h\|_{L^2(I_n)}^2 + h^{2\sigma-1}\int_{I_n}\int_{I_n}\frac{|f(x)-f(y)|^2}{|x-y|^{1+2\sigma}}\,dx\,dy\right)$$

(noting that f_h is piecewise constant). The proof can now be continued as in the last lemma. □

Proposition 3.11. *Let condition* (R) *be satisfied and assume* $\frac{1}{2} < t \leqslant r$. *Then*

$$\|f - R_h f\|_h + \|f - P_h f\|_h \leqslant Ch^t\|f\|_t \quad \text{for } f \in H^t. \tag{41}$$

Proof. Let $\sigma := \min\{t, 1\}$. First we consider the case $r \geqslant 2$, which implies $S_h^{r,1} \subset H^1 \subset H^\sigma$. Let $f_h \in S_h^{r,1}$. Then $R_h f_h = f_h$, from which follows

$$\|f - R_h f\|_h \leqslant \|f - f_h\|_h + \|R_h(f - f_h)\|_h. \tag{42}$$

For the second term we have, by the norm equivalence for spline spaces established in Proposition 3.5, together with Proposition 3.7,

$$\begin{aligned} \|R_h(f - f_h)\|_h &\leqslant C\|R_h(f - f_h)\|_0 \\ &\leqslant C\left(\|f - f_h\|_0 + \|(I - R_h)(f - f_h)\|_0\right) \\ &\leqslant C\left(\|f - f_h\|_0 + h^\sigma\|f - f_h\|_\sigma\right), \end{aligned}$$

where I is the identity operator. For the first term in (42) the same bound follows from Lemma 3.9 with f replaced by $f - f_h$. Thus we have

$$\|f - R_h f\|_h \leqslant C\left(\|f - f_h\|_0 + h^\sigma \|f - f_h\|_\sigma\right)$$

and the first result follows from the approximation power of the spline space $S_h^{r,1}$. For $r = 1$ we have $t \leqslant 1$ and hence $\sigma = t$. We may use Lemma 3.10 directly (since $R_h f \in S_h^{1,1}$), to give

$$\|f - R_h f\|_h \leqslant C\left(\|f - f_h\|_0 + h^t \|f\|_t\right)$$

and the result follows from the approximation power of the spline space $S_h^{1,1}$. The bound for $\|f - P_h f\|_h$ is proved similarly by taking P_h in place of R_h. $\square$

Since $R_h f$ involves pointwise evaluation of f, the projection R_h is not stable with respect to the norm $\|\cdot\|_0$. Nevertheless on certain subspaces we have the following relation.

Lemma 3.12. *Let condition* (R) *be satisfied, and assume* $0 \leqslant s < r - M + \frac{1}{2}$ *and* $M < r$. *Let* $\{T_h\}_{\mathcal{H}}$ *be for all* $t \in [0, r - M + \frac{1}{2})$ *a bounded sequence of mappings in* H^t. *Then the following stability estimate holds*:

$$\|R_h T_h v_h\|_s \leqslant C\|v_h\|_s \quad \text{for } v_h \in S_h^{r,M} \text{ and } h \in \mathcal{H}. \tag{43}$$

Proof. For $s > \frac{1}{2}$ we can invoke Corollary 3.8 to obtain (43). For $0 \leqslant s \leqslant \frac{1}{2}$ we choose τ, such that $\tau + s \in (\frac{1}{2}, r - M + \frac{1}{2})$ and apply Proposition 3.7 with $t = s + \tau$. Taking into account the inverse estimate for $S_h^{r,M}$ we obtain

$$\begin{aligned}\|R_h T_h v_h\|_s &\leqslant \|(I - R_h)T_h v_h\|_s + \|T_h v_h\|_s \\ &\leqslant C h^\tau \|T_h v_h\|_{s+\tau} + \|T_h v_h\|_s \\ &\leqslant C\left(h^\tau \|v_h\|_{s+\tau} + \|v_h\|_s\right) \leqslant C\|v_h\|_s. \qquad \square\end{aligned}$$

Lemma 3.13. *Let condition* (R) *be satisfied and assume* $0 \leqslant s < r - M + \frac{1}{2}$ *and* $M < r$. *Let* $g \in C^{r-M+1}(\mathbb{T})$. *Then for* $v_h \in S_h^{r,M}$ *the following convergence relation holds*:

$$v_h \to v \text{ in } H^s \text{ for } (h \in \mathcal{H}) \;\Rightarrow\; R_h(g v_h) \to g v \text{ in } H^s \text{ for } (h \in \mathcal{H}). \tag{44}$$

Proof. The operator of multiplication by the function g defines a sequence $\{T_h\}_{\mathcal{H}}$ satisfying the assumptions of Lemma 3.12. Choose $t \in (\max\{s, \frac{1}{2}\}, r - M + \frac{1}{2})$. For arbitrary $w \in H^t$ we have, taking Proposition 3.7 and Lemma 3.12 into account, the estimates

$$\begin{aligned}\|R_h(g v_h) - g v\|_s &\leqslant \|R_h g(v_h - P_h w)\|_s + \|(R_h - I)(g P_h w)\|_s + \|g(P_h w - v)\|_s \\ &\leqslant C\left(\|v_h - P_h w\|_s + h^{t-s}\|g P_h w\|_t + \|g(P_h w - v)\|_s\right) \\ &\leqslant C\left(\|v_h - P_h w\|_s + h^{t-s}\|P_h w\|_t + \|P_h w - v\|_s\right).\end{aligned}$$

Given the convergence on the left-hand side of (44) we see that lim sup $\|R_h(g v_h) - g v\|_s \leqslant C\|w - v\|_s$, yielding the assertion since H^t is dense in H^s. $\square$

The next result says that the sequence of projections R_h, $h \in \mathcal{H}$, considered as maps from the space W_h of grid functions on the mesh π_h' into H^0 is bounded.

Proposition 3.14. *Let condition* (R) *be satisfied. Then*

$$\|R_h f_h\|_0 \leqslant C \|f_h\|_h \quad \textit{for } f_h \in W_h \textit{ and } h \in \mathcal{H}. \tag{45}$$

Proof. The second estimate (33) with g replaced by f_h gives

$$\|R_h f_h\|_0^2 \leqslant C \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} |(f_h, \psi_{k,\mu})_h|^2. \tag{46}$$

We want to represent f_h with respect to the basis $\{\varphi_{j,\mu}|_{\pi'_h}, j = 1, \ldots, J, \mu \in \Lambda_h\}$ of W_h, where

$$\varphi_{j,\mu}(x) := b_{j,\mu} \sum_{v \in \Lambda_h} \Phi_\mu(vh)\gamma_j\left(\frac{x - vh}{h}\right) \quad \text{for } j = 1, \ldots, J \text{ and } \mu \in \Lambda_h.$$

Here $\gamma_j(\cdot) := \gamma(\cdot - \xi_j)$ and γ is the 1-periodically extended hat function with support $[-\varepsilon, \varepsilon]$ and $\gamma(0) = 1$, where ε is chosen small enough such that the supports of γ_j, $j = 1, \ldots, J$, do not intersect. The basis property of the $\varphi_{j,\mu}|_{\pi'_h}$ is clear since the matrix with elements $\Phi_\mu(vh)$ is nonsingular and the shifted and scaled γ_j, obviously, form a basis of W_h. The coefficients $b_{j,\mu}$ will be chosen later. A straightforward calculation shows that the Fourier series of these functions are

$$\varphi_{j,\mu} = \sum_{m \equiv \mu} \hat{\varphi}_{j,\mu}(m)\Phi_m \quad \text{for } j = 1, \ldots, J \text{ and } \mu \in \Lambda_h, \tag{47}$$

where we have used $\hat{\varphi}_{j,\mu}(m) = 0$ for $m \not\equiv \mu$ and where

$$\hat{\varphi}_{j,\mu}(m) = b_{j,\mu}\bar{\Phi}_m(h\xi_j)\varepsilon \operatorname{sinc}^2(h\varepsilon\pi m) \quad \text{for } m \equiv \mu,$$

with $\operatorname{sinc}(x) := \sin(x)/x$. We choose $b_{j,\mu}$ such that $\hat{\varphi}_{j,\mu}(\mu) = 1$, allowing (47) to be rewritten as

$$\varphi_{j,\mu}(x) = \Phi_\mu(x)\Gamma_j\left(Nx, \frac{\mu}{N}\right) \quad \text{for } j = 1, \ldots, J, \ \mu \in \Lambda_h,$$

where for $\xi \in \mathbb{R}$

$$\Gamma_j(\xi, y) := 1 + \sum_{\ell \neq 0}\left(\frac{\operatorname{sinc}(\varepsilon\pi(y + \ell))}{\operatorname{sinc}(\varepsilon\pi y)}\right)^2 \Phi_\ell(\xi - \xi_j) \quad \text{for } j = 1, \ldots, J.$$

The sum defining Γ_j converges absolutely and uniformly and $\Gamma_j(\xi, y)$ is continuous in y for $|y| \leqslant \frac{1}{2}$.

After these preparations we continue by estimating the right-hand side of (46). We represent f_h in the form

$$f_h = \sum_{j=1}^{J} \sum_{\mu \in \Lambda_h} c_{j,\mu}\varphi_{j,\mu}|_{\pi'_h} \tag{48}$$

and calculate in a similar way to (69) with the aid of the second part of Lemma A.2

$$(f_h, \psi_{k,\mu})_h = \sum_{j=1}^{J} c_{j,\mu} Q\left(\Gamma_j\left(\cdot, \frac{\mu}{N}\right), \Delta_k\left(\cdot, \frac{\mu}{N}\right)\right).$$

Since for ξ_j being a quadrature point the functions $\Delta(\xi_j, \cdot)$ are continuous on $[-\frac{1}{2}, \frac{1}{2}]$ (see [3]), we obtain

$$\sum_{\mu\in\Lambda_h} |(f_h, \psi_{k,\mu})_h|^2 \leqslant C \sum_{j=1}^{J} \sum_{\mu\in\Lambda_h} |c_{j,\mu}|^2.$$

In view of (46) it remains to prove

$$\sum_{j=1}^{J} \sum_{\mu\in\Lambda_h} |c_{j,\mu}|^2 \leqslant C \| f_h \|_h^2. \tag{49}$$

To this end, note first that, similarly to (68), we derive with the aid of Lemma A.2

$$\left\| \sum_{j=1}^{J} \sum_{\mu\in\Lambda_h} c_{j,\mu}\varphi_{j,\mu} \right\|_0^2 = \sum_{j=1}^{J} \sum_{k=1}^{J} \sum_{\mu\in\Lambda_h} G_{j,k}\left(\frac{\mu}{N}\right) c_{j,\mu}\bar{c}_{k,\mu},$$

where

$$G_{j,k}(y) := Q\big(\Gamma_j(\cdot, y), \Gamma_k(\cdot, y)\big) \quad \text{for } |y| \leqslant \tfrac{1}{2}.$$

The matrix $G(y)$ with elements $G_{j,k}(y)$ is for each y positive definite since, recalling that the $\xi_j \in [0, 1)$ are pairwise distinct, obviously the functions $\Gamma_j(\cdot, y)$ are linearly independent. Hence, since $G(y)$ is also continuous,

$$\sum_{j=1}^{J} \sum_{\mu\in\Lambda_h} |c_{j,\mu}|^2 \leqslant C \left\| \sum_{j=1}^{J} \sum_{\mu\in\Lambda_h} c_{j,\mu}\varphi_{j,\mu} \right\|_0^2. \tag{50}$$

On the other hand, the bound

$$\left\| \sum_{j=1}^{J} \sum_{\mu\in\Lambda_h} c_{j,\mu}\varphi_{j,\mu} \right\|_0^2 \leqslant C \| f_h \|_h^2, \tag{51}$$

holds since the sum in the norm is in each subinterval $[x_n, x_{n+1})$ a linear combination of the hat functions $\gamma_j((\cdot - x_n)/h)$, $j = 1, \ldots, J$, interpolating f_h in the points $x_{n,j}$, $j = 1, \ldots, J$. The estimates (50) and (51) prove (49). □

Remark 3.15. The estimate (31) can be easily derived from (45) and (41), but we preferred to give a direct proof.

4. Superapproximation and commutator properties

For the analysis of variable coefficient operator equations $Lu = f$, as in [9,18], a key role is played by certain commutator properties of the projection R_h combined with the operator of multiplication by a smooth function. Referred to in [9] as CPI, CPII, they are here called the superapproximation (Theorem 4.1) and commutator properties (Theorem 4.4).

Theorem 4.1. *Let $s_0 \in [0, r-M]$ and let $T_h : H^{s_0} \to S_h^{r,M}$, $h \in \mathcal{H}$, be projections satisfying the approximation property*

$$\|(I - T_h)f\|_{s_0} \leqslant Ch^{r-M-s_0}\|f\|_{r-M} \quad \text{for } f \in H^{r-M}. \tag{52}$$

Assume $g \in C^r(\mathbb{T})$. Then the superapproximation property

$$\|(I - T_h)(gv_h)\|_{s_0} \leqslant Ch^{1+r-M-s_0}\|g'\|_{r-1,\infty}\|v_h\|_{r-M} \quad \text{for } v_h \in S_h^{r,M} \tag{53}$$

holds true. If $s_0 \in [0, r-M)$ (and hence $M < r$) and if (52) is also satisfied with s_0 replaced by $r - M$ then

$$\|(I - T_h)(gv_h)\|_{s} \leqslant Ch^{1+t-s}\|g'\|_{r-1,\infty}\|v_h\|_{t} \quad \text{for } v_h \in S_h^{r,M}, \tag{54}$$

$s_0 \leqslant s < r - M + \frac{1}{2}$ and $t \leqslant r - M$.

The estimate (53) is called a superapproximation property, because the power of h in it is larger by 1 than the order expected from (52). Note that by Proposition 3.7 the bound (54) holds for $T_h := R_h$, the discrete orthogonal projection, with $s_0 = 0$ provided $M < r$, and thus the following corollary holds.

Corollary 4.2. *Assume $M < r$, $0 \leqslant s < r - M + \frac{1}{2}$ and $t \leqslant r - M$. Then*

$$\|(I - R_h)(gv_h)\|_{s} \leqslant Ch^{1+t-s}\|g'\|_{r-1,\infty}\|v_h\|_{t} \quad \text{for } v_h \in S_h^{r,M}. \tag{55}$$

Proof of Theorem 4.1. For given $v_h \in S_h^{r,M}$ let $w_h \in S_h^{r,M}$ be any spline satisfying the additional condition

$$\int_0^1 \big(g(x)v_h(x) - w_h(x)\big)\,dx = 0.$$

Then

$$\begin{aligned}\|(I - T_h)(gv_h)\|_{s_0} &= \|(I - T_h)(gv_h - w_h)\|_{s_0}\\ &\leqslant Ch^{r-M-s_0}\|gv_h - w_h\|_{r-M}\\ &\leqslant Ch^{r-M-s_0}\|(gv_h)^{(r-M)} - w_h^{(r-M)}\|_0,\end{aligned}$$

where in the last estimate we used the fact that for $r - M$ being a nonnegative integer the norms $\|f\|_{r-M}$ and $\|f^{(r-M)}\|_0$ are equivalent for functions having mean value zero. Note that

$$\{w_h^{(r-M)} : w_h \in S_h^{r,M}\} = S_h^{M,M},$$

i.e. equal to the space of piecewise continuous polynomials of degree $M - 1$ with breakpoints in π_h. By virtue of the approximation power of $S_h^{M,M}$ (note that this is a strictly local property, as is needed because $(gv_h)^{(r-M)}$ may have jumps at the knots) we obtain, with derivatives taken piecewise,

$$\|(gv_h)^{(r-M)} - w_h^{(r-M)}\|_0 \leqslant Ch^M\|(gv_h)^{(r)}\|_0$$

by choosing w_h adequately. We apply the Leibniz rule, take $v_h^{(r)} = 0$ (piecewise) into account and use the inverse inequality to derive further

$$\|(gv_h)^{(r-M)} - w_h^{(r-M)}\|_0$$

$$
\begin{aligned}
&\leqslant Ch^M \|g'\|_{r-1,\infty} \sum_{m=0}^{r-1} \|v_h^{(m)}\|_0 \\
&\leqslant Ch^M \|g'\|_{r-1,\infty} \left[\sum_{m=0}^{r-M} \|v_h^{(m)}\|_0 + \sum_{m=r-M+1}^{r-1} h^{r-M-m} \|v_h^{(r-M)}\|_0 \right] \\
&\leqslant Ch\|g'\|_{r-1,\infty} \|v_h\|_{r-M}. \qquad (56)
\end{aligned}
$$

This proves the first part of the theorem.

For the proof of the second part, first note that because of the inverse inequality it is sufficient to consider the case $t = r - M$ only. Since by assumption (52) holds also for s_0 replaced by $r - M$, it follows by interpolation that (52), and consequently (53), holds for s_0 replaced by s for $s \in [s_0, r - M]$. So we are left with the case $r - M < s < r - M + \frac{1}{2}$ and $t = r - M$. Here the map

$$
Df(x) := \frac{1}{i2\pi} f'(x) + \hat{f}(0),
$$

which establishes an isometric isomorphism between H^t and H^{t-1} for $t \in \mathbb{R}$, is helpful. For $v_h \in S_h^{r,M}$ and $z_h \in S_h^{M,M}$ arbitrary we obtain

$$
\begin{aligned}
\|(I - T_h)(gv_h)\|_s &= \|D^{(r-M)}[(I - T_h)(gv_h)]\|_{s-r+M} \\
&\leqslant \|D^{(r-M)}(gv_h) - z_h\|_{s-r+M} + \|D^{(r-M)}[T_h(gv_h)] - z_h\|_{s-r+M}.
\end{aligned}
$$

We apply the inverse estimate to the second term and then choose z_h to satisfy simultaneously

$$
\|D^{(r-M)}(gv_h) - z_h\|_0 \leqslant Ch^M \|[D^{(r-M)}(gv_h)]^{(M)}\|_0
$$

and

$$
\|D^{(r-M)}(gv_h) - z_h\|_{s-r+M} \leqslant Ch^{r-s} \|[D^{(r-M)}(gv_h)]^{(M)}\|_0.
$$

(The existence of such z_h has been proved for $M = 1$ in [10, Corollary 2.8] and for general M it follows from [6, Proposition A.1]; for the application of these results note that $t := s - r + M \in (0, \frac{1}{2})$ and hence the spaces H^t and $W_2^t(0, 1)$ defined in [10] are equivalent.) We obtain

$$
\begin{aligned}
&\|(I - T_h)(gv_h)\|_s \\
&\quad\leqslant C \left(h^{r-s} \|[D^{(r-M)}(gv_h)]^{(M)}\|_0 + Ch^{r-s-M} \|D^{(r-M)}[T_h(gv_h)] - z_h\|_0 \right) \\
&\quad\leqslant Ch^{r-s-M} \left(h^M \|[D^{(r-M)}(gv_h)]^{(M)}\|_0 + \|D^{(r-M)}[(1 - T_h)(gv_h)]\|_0 \right) \\
&\quad\leqslant Ch^{r-s-M} \left(h^M \|(gv_h)^{(r)}\|_0 + \|(1 - T_h)(gv_h)\|_{r-M} \right). \qquad (57)
\end{aligned}
$$

In the last estimate we used $\|[D^{(r-M)}(gv_h)]^{(M)}\|_0 \leqslant \|(gv_h)^{(r)}\|_0$, which holds true since

$$
[D^{(j)}(f)]' = \left[\frac{1}{i2\pi} \big(D^{(j-1)}(f)\big)' + \widehat{D^{(j-1)}(f)}(0) \right]' = \frac{1}{i2\pi} \big[D^{(j-1)}(f)\big]''
$$

and so forth. The first term on the right-hand side of (57) has already been estimated when working out (56). The second has already been shown to have the desired bound. Thus we obtain, for $r - M < s < r - M + \frac{1}{2}$,

$$\|(I - T_h)(gv_h)\|_s \leqslant Ch^{1+r-M-s}\|g'\|_{r-1,\infty}\|v_h\|_{r-M},$$

completing the proof. □

Remark 4.3. The result of Theorem 4.1 can be extended to projections on splines on nonuniform meshes in L_p-spaces (see [6]).

For $T_h = P_h$, the projection from (29), the inequality (54) holds with $s_0 \in (-\infty, r - M]$. In this case the result has been proved in [8], with two differences: for $s \in (r - M, r - M + \frac{1}{2})$ only the additional superconvergence power h^ρ with $\rho := r - M + 1 - s$ is obtained instead of the full power h as in (54), and the dependence of the constant on the multiplier g is not available. The commutator property CPI in the general framework of [9] contains only an additional power h^δ with some $\delta > 0$.

In [17], a commutator property is proved for the case $M = 1$ of smoothest splines with $T_h := R_h$, the qualocation projection, involving more generally a pseudo-differential operator L. If $s_0 = 0$ and $L = I$ is taken there then (54) is implied with a slightly different range of indices. The dependence of the bound on g in [17] involves derivatives of higher order than in (54).

In [9], a second kind CPII of commutator property for P_h is proved, which also contains an additional superconvergence power $o(1)$. We now give a proof of a corresponding commutator property in our context, which provides an additional power h and an explicit dependence of the constant on g.

Theorem 4.4. *Let condition* (R) *and* $M < r$ *be satisfied. Assume that* $\frac{1}{2} < t \leqslant r$, $0 \leqslant s < r - M + \frac{1}{2}$ *and* $g \in C^r(\mathbb{T})$. *Then*

$$\|R_h g(I - R_h)f\|_s \leqslant Ch^{1+t-s}\|g'\|_{r-1,\infty}\|f\|_t \quad \text{for } f \in H^t. \tag{58}$$

Proof. With the aid of Proposition 3.14 we obtain

$$\begin{aligned}
\|R_h g(I - R_h)f\|_0 &\leqslant C\|R_h g(I - R_h)f\|_h \\
&= \sup_{v_h \in S_h^{r,M}, \|v_h\|_h = 1} \big(g(I - R_h)f, v_h\big)_h \\
&= \sup_{v_h \in S_h^{r,M}, \|v_h\|_h = 1} \big((I - R_h)f, (I - R_h)(gv_h)\big)_h \\
&\leqslant \|(I - R_h)f\|_h \sup_{v_h \in S_h^{r,M}, \|v_h\|_h = 1} \|(I - R_h)(gv_h)\|_h.
\end{aligned} \tag{59}$$

The first of the last two factors can be bounded with Proposition 3.11 by

$$\|f - R_h f\|_h \leqslant Ch^t\|f\|_t.$$

Taking additionally (54) with $T_h := R_h$ and the inverse inequality into account, we obtain

$$\begin{aligned}
\|(I - R_h)(gv_h)\|_h &\leqslant Ch\|(I - R_h)(gv_h)\|_1 \leqslant Ch^2\|g'\|_{r-1,\infty}\|v_h\|_1 \\
&\leqslant Ch\|g'\|_{r-1,\infty}\|v_h\|_0 \leqslant Ch\|g'\|_{r-1,\infty}\|v_h\|_h,
\end{aligned}$$

in the last step using Proposition 3.5. The assertion now follows with another application of the inverse inequality to the left-hand side of (59). □

Remark 4.5. The same kind of reasoning, but somewhat simpler, yields the commutator property (58) with P_h in place of R_h. In this case the range of admissible indices can be increased to $0 \leqslant s \leqslant r$ (as in property CPII in [9]).

5. An example: interpolation using splines with double knots

In this section we illustrate the theory for the particular case $J = M = 2$, that is, of interpolation by double knot splines of order $r \geqslant 2$. The analysis is complete if the two interpolation points are symmetrically located (meaning that if $\xi \in (0, 1)$ is an interpolation point, then so is $1 - \xi$). In this way we extend the analysis of McLean and Prößdorf [8] for collocation with double knot splines (see [8, Section 5]), at least for the case of the identity operator, by proving a conjecture in that paper (see [8, Remark 5.5]). Similar arguments prove the conjecture for the whole class of pseudo-differential operators considered in [8].

According to Proposition 3.7 and Definition 3.6, for the case $J = M = 2$ we have stability of R_h and optimal convergence of the interpolation projection $R_h f$ to f if the condition (R) is satisfied; that is to say, if the determinant

$$F(y) := \begin{vmatrix} 1 + y^r \tilde{\Delta}_1(\xi_1, y) & \tilde{\Delta}_2(\xi_1, y) \\ 1 + y^r \tilde{\Delta}_1(\xi_2, y) & \tilde{\Delta}_2(\xi_2, y) \end{vmatrix} \tag{60}$$

is different from zero for all $y \in [-\frac{1}{2}, \frac{1}{2}]$. The easily verified symmetry property

$$F(-y) = (-1)^{r-1}\overline{F(y)} \quad \text{for } |y| < 1 \tag{61}$$

tells us that $F(0) = 0$ (and hence the property (R) fails) if F is real and r is even, or if F is imaginary and r is odd. We shall see that these two situations correspond to the two kinds of symmetric choice for the interpolation points.

From (9)–(10) we can write

$$\tilde{\Delta}_1(\xi, y) = \begin{cases} (g_r^+ + ih_r^-)(\xi, y) & \text{if } r \text{ even}, \\ (g_r^- + ih_r^+)(\xi, y) & \text{if } r \text{ odd}, \end{cases}$$

$$\tilde{\Delta}_2(\xi, y) + y\tilde{\Delta}_1(\xi, y) = \begin{cases} (g_{r-1}^- + ih_{r-1}^+)(\xi, y) & \text{if } r \text{ even}, \\ (g_{r-1}^+ + ih_{r-1}^-)(\xi, y) & \text{if } r \text{ odd}, \end{cases}$$

where for $\alpha > 0$ and $|y| < 1$

$$g_\alpha^\pm(\xi, y) := \sum_{\ell=1}^{\infty} \left[\frac{1}{(\ell + y)^\alpha} \pm \frac{1}{(\ell - y)^\alpha} \right] \cos 2\pi\ell\xi,$$

$$h_\alpha^\pm(\xi, y) := \sum_{\ell=1}^{\infty} \left[\frac{1}{(\ell + y)^\alpha} \pm \frac{1}{(\ell - y)^\alpha} \right] \sin 2\pi\ell\xi.$$

By virtue of the properties of determinants we can write

$$
\begin{aligned}
F(y) &= \begin{vmatrix} 1+y^r\tilde{\Delta}_1(\xi_1,y) & (\tilde{\Delta}_2+y\tilde{\Delta}_1)(\xi_1,y) \\ 1+y^r\tilde{\Delta}_1(\xi_2,y) & (\tilde{\Delta}_2+y\tilde{\Delta}_1)(\xi_2,y) \end{vmatrix} - y\begin{vmatrix} 1 & \tilde{\Delta}_1(\xi_1,y) \\ 1 & \tilde{\Delta}_1(\xi_2,y) \end{vmatrix} \\
&= \begin{vmatrix} 1+y^r(g_r^{\pm}+ih_r^{\mp})(\xi_1,y) & (g_{r-1}^{\mp}+ih_{r-1}^{\pm})(\xi_1,y) \\ 1+y^r(g_r^{\pm}+ih_r^{\mp})(\xi_2,y) & (g_{r-1}^{\mp}+ih_{r-1}^{\pm})(\xi_2,y) \end{vmatrix} \\
&\quad -y\big(\tilde{\Delta}_1(\xi_2,y)-\tilde{\Delta}_1(\xi_1,y)\big) \\
&= \begin{vmatrix} (k_r^{\pm}+iy^rh_r^{\mp})(\xi_1,y) & (g_{r-1}^{\mp}+ih_{r-1}^{\pm})(\xi_1,y) \\ (k_r^{\pm}+iy^rh_r^{\mp})(\xi_2,y) & (g_{r-1}^{\mp}+ih_{r-1}^{\pm})(\xi_2,y) \end{vmatrix} \\
&\quad -y\big((g_r^{\pm}+ih_r^{\mp})(\xi_2,y)-(g_r^{\pm}+ih_r^{\mp})(\xi_1,y)\big),
\end{aligned}
$$

where the upper or lower signs hold according to whether r is even or odd, and

$$k_r^{\pm}(\xi,y) := 1+y^r g_r^{\pm}(\xi,y).$$

The next two propositions cover the two possible kinds of symmetrically arranged interpolation points. The first proposition is a special case of McLean and Prößdorf [8, Lemma 5.1]. The first part of the second proposition is a conjecture in [8].

Proposition 5.1. *Let $J = M = 2$, and let $\xi_1 = 0$ and $\xi_2 = \frac{1}{2}$. If r is even then $F(0) = 0$, and the condition* (R) *fails. If r is odd then the condition* (R) *holds, and the interpolant $R_h f \in S_h^{r,M}$ satisfies*

$$\|R_h f - f\|_s \leqslant Ch^{t-s}\|f\|_t \quad \text{for } f \in H^t \tag{62}$$

and all s, t satisfying $0 \leqslant s < r - M + \frac{1}{2},\ \frac{1}{2} < t \leqslant r$.

Proof. Noting that

$$h_\alpha^{\pm}(\xi,y) = 0 \quad \text{if } \xi = 0 \text{ or } \tfrac{1}{2},$$

we see with this choice $F(y)$ is real. If r is even then (61) gives $F(0) = 0$, and condition (R) fails.

If r is odd and $\xi_1 = 0$, $\xi_2 = \frac{1}{2}$ we have

$$F(y) = \begin{vmatrix} k_r^-(0,y) & g_{r-1}^+(0,y) \\ k_r^-(\frac{1}{2},y) & g_{r-1}^+(\frac{1}{2},y) \end{vmatrix} - y\big(g_r^-(\tfrac{1}{2},y) - g_r^-(0,y)\big).$$

Since by (61) F is now an even function, it is sufficient to consider $y \in [0, \frac{1}{2}]$. But for $y \in [0, \frac{1}{2}]$ it is shown in [3, Theorem 2] that for $\alpha > 0$

$$g_\alpha^-(\tfrac{1}{2},y) > 0, \quad g_\alpha^-(0,y) < 0, \quad g_\alpha^+(\tfrac{1}{2},y) < 0, \quad g_\alpha^+(0,y) > 0$$

and

$$k_\alpha^{\pm}(\xi,y) \geqslant 0, \quad k_\alpha^+(\xi,y) > 0 \quad \text{for } \xi \neq \tfrac{1}{2}, \quad k_\alpha^-(\xi,y) > 0 \quad \text{for } \xi \neq 0 \tag{63}$$

from which $F(y) < 0$ follows immediately. □

Proposition 5.2. *Let $J = M = 2$, and let $\xi_1 = \varepsilon,\ \xi_2 = 1 - \varepsilon$ with $\varepsilon \in (0, \frac{1}{2})$. If r is even then the condition* (R) *holds, and the interpolant $R_h f \in S_h^{r,M}$ satisfies* (62). *If r is odd then $F(0) = 0$, thus the condition* (R) *fails.*

Proof. Since $\tilde{\Delta}_k(1-\varepsilon, y) = \overline{\tilde{\Delta}_k(\varepsilon, y)}$, it follows from (60) that the choice $\xi_1 = \varepsilon,\ \xi_2 = 1-\varepsilon$ makes F imaginary. It then follows from (61) that $F(0) = 0$ if r is odd, thus condition (R) fails in this case. If r is even then F is even, thus it is sufficient to consider $y \in [0, \frac{1}{2}]$. In this case, noting that $g^{\pm}_\alpha(1-\varepsilon, \cdot) = g^{\pm}_\alpha(\varepsilon, \cdot)$, $h^{\pm}_\alpha(1-\varepsilon, \cdot) = -h^{\pm}_\alpha(\varepsilon, \cdot)$ and $k^{\pm}_\alpha(1-\varepsilon, \cdot) = k^{\pm}_\alpha(\varepsilon, \cdot)$,

$$\begin{aligned} F(y) &= 2i\left(-k^+_r(\varepsilon, y)h^+_{r-1}(\varepsilon, y) + y^r h^-_r(\varepsilon, y)g^-_{r-1}(\varepsilon, y) + yh^-_r(\varepsilon, y)\right) \\ &= 2i\left(-k^+_r(\varepsilon, y)h^+_{r-1}(\varepsilon, y) + yh^-_r(\varepsilon, y)k^-_{r-1}(\varepsilon, y)\right). \end{aligned}$$

That the expression in brackets is nonpositive follows from (63) and from (see [3, Theorem 2])

$$h^+_\alpha(\varepsilon, y) \geqslant 0, \quad h^-_\alpha(\varepsilon, y) \leqslant 0 \quad \text{for } y \in [0, \tfrac{1}{2}].$$

The proof of Brown et al. [3, Theorem 2] also yields the stronger result that $h^-_\alpha(\varepsilon, y)$ is negative for $\varepsilon \in (0, \frac{1}{2})$, proving that the expression in the brackets is positive, and completing the proof. □

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Appendix A.

This appendix provides a collection of simple results invoked in the rest of the paper.

Lemma A.1.

$$\sum_{n=0}^{N-1} \Phi_{\ell-m}(nh) = \begin{cases} 0 & \text{if } \ell \not\equiv m, \\ N & \text{if } \ell \equiv m \end{cases} \tag{64}$$

and

$$(\Phi_\ell, \Phi_m)_h = \begin{cases} 0 & \text{if } \ell \not\equiv m, \\ Q(\Phi_{\frac{\ell-m}{N}}, 1) & \text{if } \ell \equiv m. \end{cases} \tag{65}$$

Proof. The first result is trivial, given

$$\sum_{n=0}^{N-1} \Phi_{\ell-m}(nh) = \sum_{n=0}^{N-1} \left(e^{2\pi i(\ell-m)/N}\right)^n.$$

The second follows from

$$\begin{aligned} (\Phi_\ell, \Phi_m)_h &= \sum_{n=0}^{N-1} h \sum_{j=1}^{J} \omega_j \Phi_{\ell-m}(x_n + h\xi_j) \\ &= h \sum_{n=0}^{N-1} \Phi_{\ell-m}(nh) \sum_{j=1}^{J} \omega_j \Phi_{\frac{\ell-m}{N}}(\xi_j). \qquad \square \end{aligned}$$

Many integrals of interest in this paper take the form considered in the next lemma.

Lemma A.2. *Let D be a 1-periodic function on $\mathbb{R}$. If $D \in L^1(\mathbb{T})$ then*

$$\int_0^1 \Phi_{\ell-m}(x) D(Nx)\, dx = \begin{cases} 0 & \text{if } \ell \not\equiv m, \\ \int_0^1 \Phi_{\frac{\ell-m}{N}}(\xi) D(\xi)\, d\xi & \text{if } \ell \equiv m. \end{cases} \tag{66}$$

If D is defined everywhere in $\mathbb{R}$ then

$$Q_N\big(\Phi_{\ell-m} D(N\cdot)\big) = \begin{cases} 0 & \text{if } \ell \not\equiv m, \\ Q_1\big(\Phi_{\frac{\ell-m}{N}} D\big) & \text{if } \ell \equiv m. \end{cases} \tag{67}$$

Proof. The left-hand side of the first identity can be written as

$$\begin{aligned} \int_0^1 \Phi_{\ell-m}(x) D(Nx)\, dx &= \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} \Phi_{\ell-m}(x) D(Nx)\, dx \\ &= h \sum_{n=0}^{N-1} \int_0^1 \Phi_{\ell-m}(x_n + h\xi) D(n+\xi)\, d\xi \\ &= \left(h \sum_{n=0}^{N-1} \Phi_{\ell-m}(nh) \right) \int_0^1 \Phi_{\frac{\ell-m}{N}}(\xi) D(\xi)\, d\xi, \end{aligned}$$

where in the last step we used the periodicity of D. The first identity then follows from (64).

The second identity follows in almost the same way, if we recall that the rule Q_N is an n-fold copy of the rule Q_1: we have

$$\begin{aligned} Q_N\big(\Phi_{\ell-m} D(N\cdot)\big) &= h \sum_{n=0}^{N-1} Q_1\big(\Phi_{\frac{\ell-m}{N}}(n+\cdot) D(n+\cdot)\big) \\ &= \left(h \sum_{n=0}^{N-1} \Phi_{\ell-m}(nh) \right) Q_1\big(\Phi_{\frac{\ell-m}{N}} D\big). \qquad \Box \end{aligned}$$

Corollary A.3. *For $\mu, \nu \in \Lambda_h$ and $k, \ell = 1, \dots, M$*

$$(\psi_{k,\mu}, \psi_{\ell,\nu})_0 = \delta_{\mu\nu} \left(\Delta_k\left(\cdot, \frac{\mu}{N}\right), \Delta_\ell\left(\cdot, \frac{\nu}{N}\right) \right)_0, \tag{68}$$

$$(\psi_{k,\mu}, \psi_{\ell,\nu})_h = \delta_{\mu\nu} Q \left(\Delta_k\left(\cdot, \frac{\mu}{N}\right), \Delta_\ell\left(\cdot, \frac{\nu}{N}\right) \right). \tag{69}$$

Proof. Using the representation (13) for $\psi_{k,\mu}$ we obtain

$$(\psi_{k,\mu}, \psi_{\ell,\nu})_0 = \int_0^1 \Phi_{\mu-\nu}(x) \Delta_k\left(Nx, \frac{\mu}{N}\right) \overline{\Delta}_\ell\left(Nx, \frac{\nu}{N}\right) dx$$

and the first result follows from the identity in Lemma A.2 on setting $D := \Delta_k(\cdot, \frac{\mu}{N})\overline{\Delta}_\ell(\cdot, \frac{\nu}{N})$, and noting the obvious 1-periodicity of $\Delta_k(\cdot, y)$. The second result follows similarly from the second identity in Lemma A.2. $\Box$

The following result is also an immediate consequence of Lemma A.2.

Corollary A.4. *For $\mu \in \Lambda_h$, $k = 1, \ldots, M$ and $\ell \in \mathbb{Z}$*

$$(\Phi_\ell, \psi_{k,\mu})_0 = \begin{cases} 0 & \text{if } \ell \not\equiv \mu, \\ \left(\Phi_{\frac{\ell-\mu}{N}}, \Delta_k(\cdot, \frac{\mu}{N})\right)_0 & \text{if } \ell \equiv \mu, \end{cases} \tag{70}$$

$$(\Phi_\ell, \psi_{k,\mu})_h = \begin{cases} 0 & \text{if } \ell \not\equiv \mu, \\ Q\left(\Phi_{\frac{\ell-\mu}{N}}, \Delta_k(\cdot, \frac{\mu}{N})\right) & \text{if } \ell \equiv \mu. \end{cases} \tag{71}$$

Lemma A.5. *The Fourier series of the spline functions $\psi_{k,\mu}$ are given by*

$$\psi_{1,\mu}(x) = \Phi_\mu(x) + \sum_{m\equiv\mu}{}' \left(\frac{\mu}{m}\right)^r \Phi_m(x), \tag{72}$$

$$\psi_{k,\mu}(x) = \sum_{m\equiv\mu}{}' \left(\frac{N}{m}\right)^r \left(\frac{m-\mu}{N}\right)^{k-1} \Phi_m(x) \quad \text{for } k = 2, \ldots, M, \tag{73}$$

where the prime on the sum indicates that the $m = \mu$ term is to be omitted.

Proof. From (13), (11) and (9) we can write

$$\begin{aligned}\psi_{1,\mu}(x) &= \Phi_\mu(x) + \mu^r \sum_{\ell\neq 0} \frac{1}{(\mu+\ell N)^r} \Phi_{\mu+\ell N}(x) \\ &= \Phi_\mu(x) + \sum_{m\equiv\mu}{}' \left(\frac{\mu}{m}\right)^r \Phi_m(x),\end{aligned}$$

while (12) gives for $k = 2, \ldots, M$

$$\begin{aligned}\psi_{k,\mu}(x) &= N^r \sum_{\ell\neq 0} \frac{\ell^{k-1}}{(\mu+\ell N)^r} \Phi_{\mu+\ell N}(x) \\ &= \sum_{m\equiv\mu}{}' \left(\frac{N}{m}\right)^r \left(\frac{m-\mu}{N}\right)^{k-1} \Phi_m(x). \qquad \square\end{aligned}$$

Lemma A.6. *Let $v_h \in S_h^{r,M}$ and $s < r - M + \frac{1}{2}$. If v_h has the representation*

$$v_h = \sum_{k=1}^{M} \sum_{\mu\in\Lambda_h} c_{k,\mu}\psi_{k,\mu} \tag{74}$$

then

$$\|v_h\|_s^2 = \sum_{\mu\in\Lambda_h} \langle\mu\rangle^{2s} |c_{1,\mu}|^2 + \sum_{\mu\in\Lambda_h} \left\| \sum_{k=1}^{M} \tilde{c}_{k,\mu}\Omega_k^{(s)}\left(\cdot, \frac{\mu}{N}\right) \right\|_0^2, \tag{75}$$

where

$$\tilde{c}_{k,\mu} := \begin{cases} \mu^r N^{s-r} c_{1,\mu} & \text{for } k = 1, \\ N^s c_{k,\mu} & \text{for } k = 2, \ldots, M \end{cases} \tag{76}$$

and

$$\Omega_k^{(s)}(\xi, y) := \sum_{\ell \neq 0} |y+\ell|^s \frac{\ell^{k-1}}{(y+\ell)^r} \Phi_\ell(\xi) \quad \textit{for } \xi \in \mathbb{R} \textit{ and } |y| \leqslant \frac{1}{2}. \tag{77}$$

Proof. For the proof it is helpful to introduce the following linear operator A_s (more precisely, a pseudo-differential operator of order s) defined by its effect on the Fourier series of the distribution f:

$$f(x) = \sum_{\ell=-\infty}^{\infty} \hat{f}(\ell)\Phi_l(x) \to A_s f(x) = \sum_{\ell=-\infty}^{\infty} \langle \ell \rangle^s \hat{f}(\ell)\Phi_l(x).$$

One has

$$\|f\|_s^2 = \sum_{\ell=-\infty}^{\infty} \langle \ell \rangle^{2s} |\hat{f}(\ell)|^2 = \|A_s f\|_0^2. \tag{78}$$

We apply this relation to v_h in the form (74). Taking the Fourier representations (72) and (73) into account, we obtain

$$\begin{aligned}
A_s v_h(x) &= \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} c_{k,\mu} A_s \psi_{k,\mu}(x) \\
&= \sum_{\mu \in \Lambda_h} c_{1,\mu} \left[\langle \mu \rangle^s \Phi_\mu(x) + \sum_{m \equiv \mu}{}' \left(\frac{\mu}{m}\right)^r |m|^s \Phi_m(x) \right] \\
&\quad + \sum_{k=2}^{M} \sum_{\mu \in \Lambda_h} c_{k,\mu} \sum_{m \equiv \mu}{}' \left(\frac{N}{m}\right)^r \left(\frac{m-\mu}{N}\right)^{k-1} |m|^s \Phi_m(x) \\
&= \sum_{\mu \in \Lambda_h} c_{1,\mu} \langle \mu \rangle^s \Phi_\mu(x) + \sum_{\mu \in \Lambda_h} \tilde{c}_{1,\mu} \sum_{\ell \neq 0} \frac{|\mu/N + \ell|^s}{(\mu/N+\ell)^r} \Phi_{\mu+N\ell}(x) \\
&\quad + \sum_{\mu \in \Lambda_h} \sum_{k=2}^{M} \tilde{c}_{k,\mu} \sum_{\ell \neq 0} \frac{\ell^{k-1} |\mu/N+\ell|^s}{(\mu/N+\ell)^r} \Phi_{\mu+N\ell}(x) \\
&= \sum_{\mu \in \Lambda_h} c_{1,\mu} \langle \mu \rangle^s \Phi_\mu(x) + \sum_{\mu \in \Lambda_h} \Phi_\mu(x) \sum_{k=1}^{M} \tilde{c}_{k,\mu} \Omega_k^{(s)} \left(Nx, \frac{\mu}{N}\right).
\end{aligned}$$

We now invoke (78), and prove (75) by noting that the last two sums are orthogonal in H^0 and that

$$\int_0^1 \Phi_{\mu-\nu}(x) \Omega_k^{(s)} \left(Nx, \frac{\mu}{N}\right) \overline{\Omega_j^{(s)}} \left(Nx, \frac{\nu}{N}\right) dx = \delta_{\mu\nu} \int_0^1 \Omega_k^{(s)} \left(\xi, \frac{\mu}{N}\right) \overline{\Omega_j^{(s)}} \left(\xi, \frac{\nu}{N}\right) d\xi,$$

which follows immediately from Lemma A.2. $\square$

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